PROBLEM SOLVING/DOUBT CLEARING SESSION-1

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SOLUTUIONS

1.Let $x = \frac{a}{b}, y = \frac{b}{c}$ and $z = \frac{c}{a}$ So,

$$P = \frac{\frac{a}{b}+1}{\frac{a}{c}+\frac{a}{b}+1} + \frac{\frac{b}{c}+1}{\frac{b}{a}+\frac{b}{c}+1} + \frac{\frac{c}{a}+1}{\frac{c}{b}+\frac{c}{a}+1}$$

$$\implies P = \frac{bc+ca}{ab+bc+ca} + \frac{ab+ca}{ab+bc+ca} + \frac{ab+bc}{ab+bc+ca}$$

$$\implies P = \frac{2(ab+bc+ca)}{ab+bc+ca} = 2$$

2.Let $f(n)$ be the number of length n binary strings with $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $f(3) = 6$. To find $f(n+1)$, we can append a 0 or a 1 to any of the $f(n)$ length-n binary strings, then subtract off the cases where the digits in the $n-1$, n, and $n+1$th positions are equal. This should happen in $f(n-2)$ ways since if the last three digits are 1, then the first $n-2$ digits can be any valid binary string ending in 0, and if the last three digits are 0, the first $n-2$ digits can be any valid string ending in 1.

Then the recurrence is $f(n+1) = 2f(n) - f(n-2)$. We can either solve the recursion or just plug and chug to get $f(10) = 178$. (It also happens that $f(n+1) = f(n) + f(n-1)$).

3.Let there be n players and we are asked to build two teams comprising of two players in each team.But thease two teams can have one player in common or all the four players are distinct.So,we can do this in $\binom{\binom{n}{2}}{2}$ways.

Clearly,the above argument suffices the LHS.

Now,let us derive the RHS.

First of all we will look into the case where two teams have one player in common.

So,for this case we will select 3 players out of n players which we can do in $\binom{n}{3}$ ways.Now to determine the player who gets repeated,we could to do that in $\binom{3}{1}\binom{n}{3} = 3\binom{n}{3}$ways.

Now,we would look into case where all the four players are distinct.

So,for this case we will select 4 players out n players,which we can do in $\binom{n}{4}$ways and each such group of 4 players can be split into two teams in $\frac{4!}{(2!)^2 2!}\binom{n}{4} = \frac{4!}{(2!)^3}\binom{n}{4} = \frac{24}{8}\binom{n}{4} = 3\binom{n}{4}$ ways.

So,adding the above two cases we get,$3\left(\binom{n}{3} + \binom{n}{3}\right) = 3\binom{n+1}{4}$ which suffices the RHS.

Thus,$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$

So,we can write

$$\sum_{k=3}^{x} \binom{\binom{k}{2}}{2} = 3\sum_{k=3}^{x} \binom{k+1}{4}$$

So, we get

$$3\sum_{k=3}^{x} \binom{k+1}{4} \leq 168 \implies \sum_{k=3}^{x} \binom{k+1}{4} \leq 56$$

Now,$\sum_{k=3}^{x} \binom{k+1}{4} = \binom{4}{4} + \binom{5}{4} + \cdots + \binom{x+1}{4}$

Claim :$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$

Proof:We know,that $\binom{m}{r} = \binom{m+1}{r+1} - \binom{m}{r+1}$ Now put $m = r, r+1, r+2, \ldots, n$ and adding we will get,

$$\begin{aligned}
\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1} = &\binom{r+1}{r+1} - \binom{r}{r+1} \\
&+\binom{r+2}{r+1} - \binom{r+1}{r+1} \\
&+\binom{r+3}{r+1} - \binom{r+2}{r+1} \\
&\quad\vdots \\
&+\binom{n+1}{r+1} - \binom{n}{r+1} \\
&-------------- \\
\binom{n+1}{r} - &\binom{r}{r+1} = \binom{n+1}{r+1}
\end{aligned}$$

Thus,

$$\sum_{k=3}^{x} \binom{k+1}{4} = \binom{x+2}{5} \leq 56$$

On solving we get, $x = 3, 4, 5, 6$.

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